Suggested solution of HW1

P.171 Q4:

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then $f(x) - f(0) = \varphi(x)(x - 0)$, for all $x \in \mathbb{R}$.
Since $|\varphi(x)| \le |x|$, by squeeze theorem, $\varphi(x)$ is continuous at $x = 0$. By Carathéodory's Theorem, f is differentiable at $x = 0$ and $f'(0) = \varphi(0) = 0$.

P.171 Q10:

At $c \neq 0$, the function $f(x) = \dfrac{1}{x^2}$ is differentiable at c and the function $h(x) = \sin x$ is differentiable at $\dfrac{1}{c^2}$. By Chain Rule, $h \circ f$ is differentiable at c and

$$(g)'(c) = h'(f(c))f'(c) = 2c \sin \frac{1}{c^2} - \frac{2}{c} \cos \frac{1}{c^2}.$$

At c=0,

$$\left| \frac{g(x) - g(0)}{x} \right| = \left| x \sin \frac{1}{x^2} \right| \le |x|$$

let $\epsilon > 0$, there exists $\delta = \epsilon > 0$ such that

$$\left| \frac{g(x) - g(0)}{x} \right| \le |x| < \epsilon, \ \forall\ 0 < |x - 0| < \delta.$$

So g is differentiable for all $x \in \mathbb{R}$.

To show that g' is unbounded, we pick a sequence $\{x_n\}$ such that $x_n = \dfrac{1}{2\pi n}$, $\forall n \in \mathbb{N}$.
Then, $g'(x_n) = -4\pi n$ which is unbounded.

P.179 Q5:

n is a given natural number. Let $f : [1, +\infty) \to \mathbb{R}$ by $f(x) = x^{\frac{1}{n}} - (x - 1)^{\frac{1}{n}}$.

$$f'(x) = \frac{1}{n} \left[x^{\frac{1-n}{n}} - (x-1)^{\frac{1-n}{n}}\right] < 0 \ , \ \forall x \ge 1.$$

So by mean value theorem, for all $x > y \ge 1$, there exists $\eta \in (y, x)$ such that

$$f(x) - f(y) = f'(\eta)(x - y) < 0 \ , \ \forall\ 1 \le y < x.$$

If $a > b > 0$, put $x = \frac{a}{b}$, $y = 1$, it implies

$$1 = f(1) > f(\frac{a}{b}) = (\frac{a}{b})^{\frac{1}{n}} - (\frac{a}{b} - 1)^{\frac{1}{n}}$$

$$\implies (a - b)^{\frac{1}{n}} > a^{\frac{1}{n}} - b^{\frac{1}{n}}.$$

P.179 Q7:

Let $f:(0,+\infty)\to\mathbb{R}$ by $f(x)=\log x$.

By mean value theorem, for any $x>1$, there exists $\eta\in(1,x)$ such that

$$\log x = f(x)-f(1)=f'(\eta)(x-1)=\frac{1}{\eta}(x-1).$$

Since $\eta\in(1,x)$, $\dfrac{1}{x}<\dfrac{1}{\eta}<1$. Therefore,

$$\frac{x-1}{x}<\log x<x-1,\ \forall\, x>1.$$